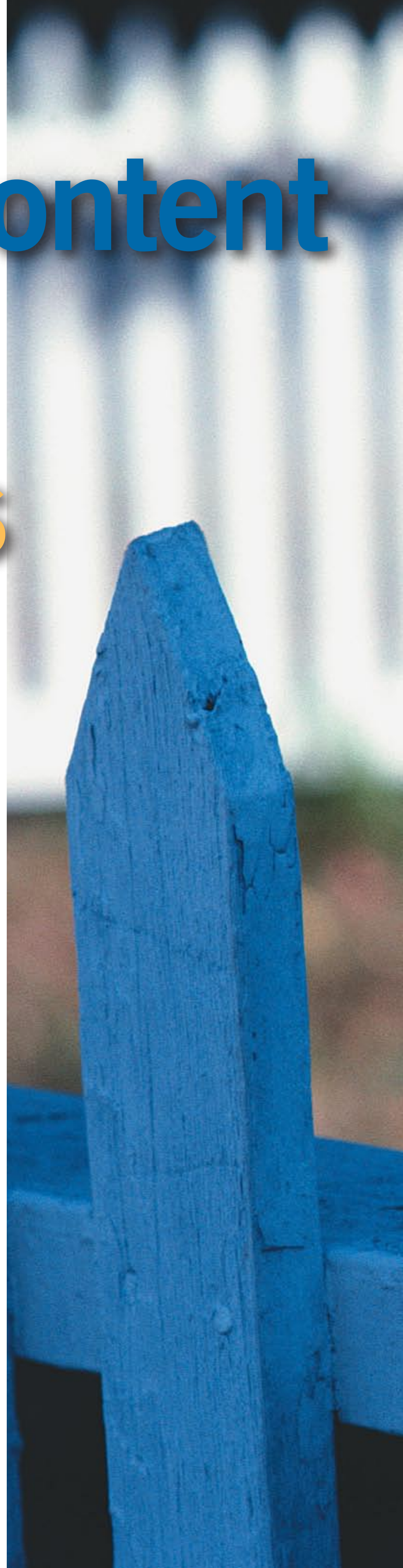
Integrating Content to Create Problem-Solving Opportunities

Darin Beigie



> Problem solving is not a distinct topic, but a process that should permeate the study of mathematics and provide a context in which concepts and skills are learned. (NCTM 2000, p. 182)

Advocates for problem-centered mathematics teaching argue that problem solving is not only an essential skill, it is also intertwined with a robust and flexible understanding of mathematical content (Hiebert and Wearne 2003; Lindquist 1989; Schroeder and Lester 1989; Stein et al. 2000). Mathematics teaching in the United States, however, has had a varied commitment to learning through problem solving, as reflected in the large number of traditional exercises that can still be found in some textbooks. Faced with the expectation of content and skill mastery in conventional test situations, some teachers, students, and parents balk at a learning environment that strays too much from direct instruction followed by repetitive practice. Good examples of the ongoing dialogue about a problem-centered mathematics curriculum can be found in Bay-Williams and Meyer (2005), Burns (1992), Lubienski (1999), and Sawada (1999).

This article describes one seventh-grade teacher's classroom efforts to integrate traditional exercises from different content areas to form more robust questions that provide genuine problem-solving opportunities for students. Since the problems are generally the result of combining two specific procedures from different content areas, they can be viewed as perhaps closely related to traditional exercises. However, the need to discern and combine distinct techniques from different content areas usually presents students with a challenging problem-solving situation. Such experiences not only foster higher-order thinking skills among the students, they also lead to a deeper understanding and a better mastery of mathematical content.

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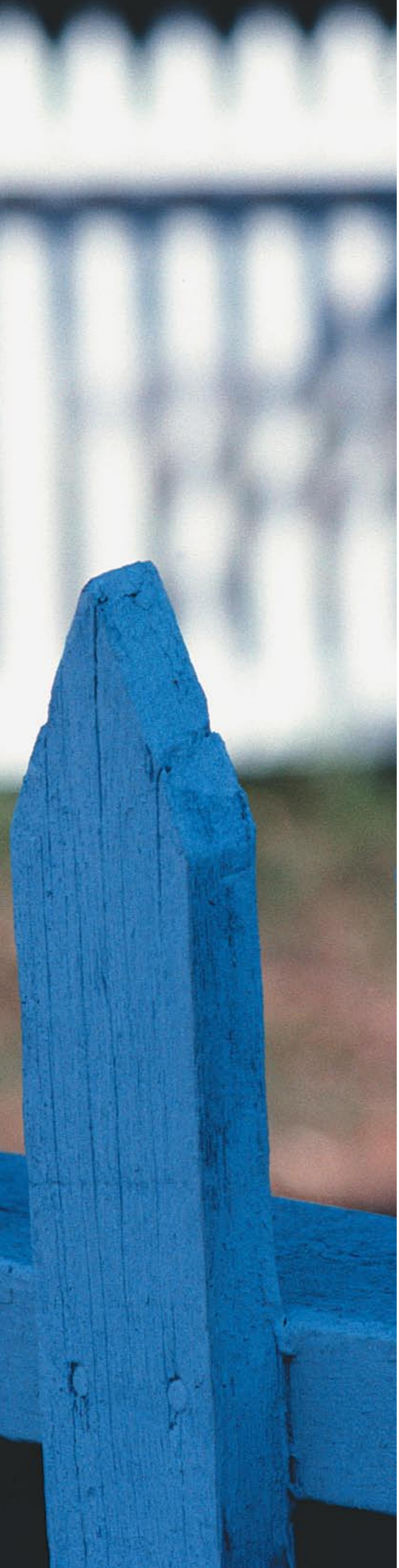


PERSPECTIVES ON PROBLEM SOLVING

The difference between an exercise and a problem can be illustrated by posing different questions about the topic of perimeter. An *exercise* practices computation from a learned algorithm. See this example.

> What is the perimeter of a rectangle that is 12 yards long and 4 yards wide?

A *problem* incorporates some critical thinking into the question. See the following example.

> What are the possible perimeters of a rectangle with an area of 48 square yards?

Contextualizing the problem can be helpful, such as having the perimeter represent a wooden fence around a rectangular garden. Adding a goal of optimization makes the problem even more robust, as the following shows.

> What is the least amount of wooden fencing needed for the perimeter of a rectangular garden with an area of 48 square yards?

For a working definition of what is meant by a problem, as opposed to an exercise, simple statements are most effective: "A problem is a situation in which a person is seeking some goal and for which a suitable course of action is not immediately apparent" (Burns 1992, p. 16). A mathematical question may be sophisticated, but it is not a problem unless the student makes real decisions about how to construct the solution. What constitutes a problem thus depends not only on the question but also on the context in which it is solved.

Pólya (1957) offered what is now a classic four-step framework for solving problems: understand the problem, make a plan, execute the plan, and check your answer. The second step, make a plan, is a key ingredient in the problem-solving experience, one that distinguishes a problem from an exercise. Stein et al. (2000) emphasize the notion of cognitive demands associated with mathematical tasks. The lower-level cognitive demands associated with exercises often involve "exact reproduction of previously seen material and what is to be reproduced is clearly and directly stated" (p. 16), whereas the higher-level cognitive demands associated with problem solving "require students to explore and understand the nature of mathematical concepts, processes, or relationships" (p. 16).

By its very nature, mathematics instruction through problem solving means that students take greater ownership in making decisions and constructing answers in the face of mathematical questions. The teacher must be willing to play a less direct role in the instruction and allow students enough classroom time to "grapple with problems, search for strategies and solutions on their own, and learn to evaluate their own results" (Burns 1992, p. 29). The classroom tone must encourage independence, risk-taking, and perseverance. Indeed, problem solving should not be viewed as purely an intellectual affair, for "determination and emotions play an important role" (Pólya 1957, p. 93).

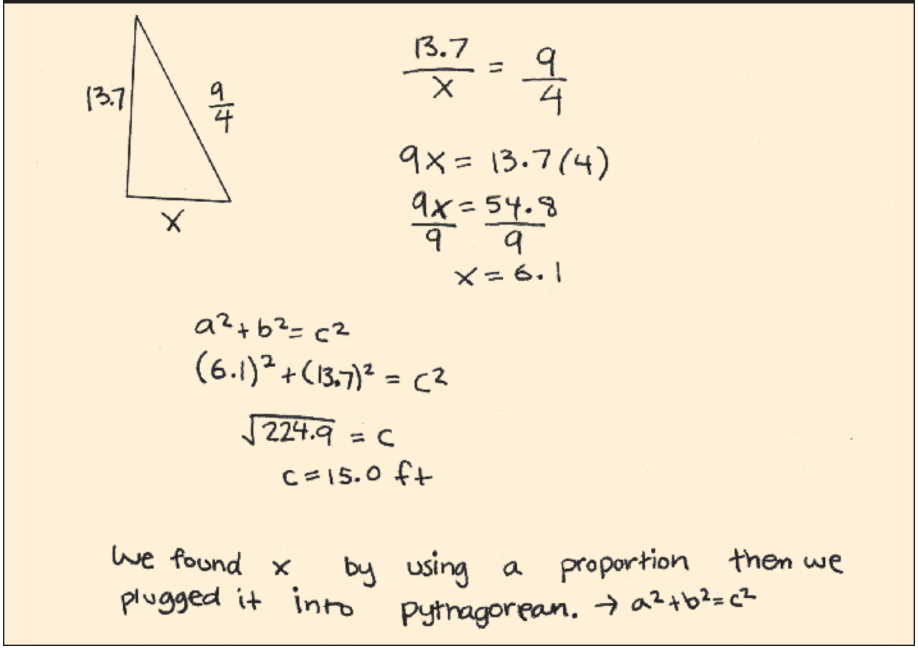
In contrasting the critical thinking of problem solving with the repetitive work of traditional exercises, we do not wish to convey that only problem solving has merit. Indeed, direct instruction followed by repetitive practice is a healthy component of instruction in many endeavors, such as sports and music. Even at a professional level, inspired coaches, gifted athletes, and brilliant musicians all have repetitive practice as some part of their foundation. Mathematics

Fig. 1 Leaning Ladder problem

Problem 1. A ladder leans against a vertical wall at a slope of 9/4. The top of the ladder is 13.7 feet from the ground. What is the length of the ladder?

Content connection: Ratios

Fig. 2 This is a correct student solution to the Leaning Ladder problem. The 9/4 label on the triangle's hypotenuse is understood by the student to represent the tilt of the ladder, not its length.



classes that get too drawn into direct instruction and traditional exercises, however, can be like a sports team that never plays a game or a musician who never performs a recital.

INTEGRATING CONTENT TO CREATE A PROBLEM-SOLVING OPPORTUNITY

Seventh-grade prealgebra students are young enough to study a rich variety of content in general mathematics, yet old enough to tackle these subjects with some depth and some ability to make connections. There can be lost opportunities when prealgebra textbooks compartmentalize content primarily into separate chapters, with little overlap from one chapter to the next. Most questions, whether exercises or problems, tend to be framed within the content of the chapter. A goal for the prealgebra classes I teach is to have students master a concept or procedure in a way that exhibits flexibility and robustness; whenever possible, I supplement lessons with questions that connect content and techniques from throughout the year. These connections turn traditional exercises into problem-solving opportunities.

To illustrate, we present a collection of problems that connect the Pythagorean theorem with a variety of topics throughout the prealgebra curriculum. These problems have been incorporated in my prealgebra classes; sometimes they are grouped together over the course of a few days, sometimes scattered as individual questions throughout the year. Such problems are usually worked on by students in pairs or triplets. We discuss one problem in detail in this section, followed by a more streamlined presentation of later problems in the next section. Each question is accompanied by a reference to the content connected with the Pythagorean theorem, something that is not included in the student questions.

A traditional Pythagorean question would be to give two legs of a right triangle and ask students to calculate the hypotenuse. Suppose instead we give one leg and the slope of the hypotenuse, as in the Leaning Ladder problem (see **fig. 1**). The solution to this problem is a straightforward combination of two procedures: (1) use the slope ratio to make a proportion equation to solve for the missing leg in the right triangle formed by the ladder, the wall, and the ground; then (2) use the Pythagorean theorem to calculate the length of the ladder from the two legs. An example of a correct student solution is shown in **figure 2**. Each separate procedure forms the basis of a traditional exercise that is quite manageable for prealgebra students—a standard proportion calculation and a routine Pythagorean application. When the procedures are brought together in one question, however, the result is a nonroutine problem for most students. It is important to note how the question does not exhibit some trademark properties often associated with more involved problem-solving situations, such as multiple answers and a variety of solution methods. The problem is not open-ended—there is one answer and arguably one efficient solution method. The question nonetheless is very much a problem, for it requires students to make real decisions about their plan for solution—to draw together two procedures from separate content topics, learned at substantially different times in the year.

To solve the Leaning Ladder problem, most students draw a diagram and understand what length needs to be calculated. Those students who pick up quickly on the role of slope in making a proportion solve the question easily in a few minutes. The majority of students, however, need several minutes to sort things out. Faced with multiple components in the problem, for some students it is as though all but one of the components is invisible. These students typically lock into the Pythagorean part of the problem and state things like "how do you get the hypotenuse if you only have the length of one side?" Some students realize that the ladder's slope has relevance to the missing length but are not sure how to apply the concept of slope outside of a graph. Drawing on the work of Garofalo and Lester (1985) and Schoenfeld (1987), NCTM (2000) observes that "students' problem-solving failures are often due not to a lack of mathematical knowledge but to the ineffective use of what they do know" (p. 54). The students who struggle with the slope section of the problem would be able to calculate the slope of a linear graph, but they have difficulty transferring this knowledge to a tilted ladder. Posing clarifying questions can help students make this transference, and they ask, "Does the slope mean for the ladder that for every 4 feet you go over, you go up 9 feet?" For students who struggle with the problem, the "a ha!" moment is visible on their faces when they recognize that they can use slope to make a proportion and find a missing length.

When students are less accustomed to a problem-solving situation in which they need to connect distinct procedures, some can be uneasy with the process and want to give up. In these instances, I hear "this one's impossible" or "we can't get this one." These responses are in keeping with the NCTM (2000) observations that "many students have developed the faulty belief that all mathematics problems could be solved quickly and directly. If they do not immediately know how to solve a problem, they will give up, which supports a view of themselves as incompetent problem solvers" (p. 259). I ask the students to give the problem more time, to keep talking things over with their partners. Indeed, the majority of students who are initially stuck with the problem are able to find the correct solution through persistence and cooperation.

In general, the students who show greater authority with traditional exercises are more proficient at combining distinct procedures to solve a problem, but there can be notable exceptions at either end. They also exhibit an interesting equalizing tendency. Students who struggle more with traditional content tend to persevere in the face of difficulties, a trait that sometimes leads to stronger results in a challenging situation. By contrast, some students who work very fast on traditional exercises will freeze when faced with a situation in which a prescribed algorithm has not been presented to them. Pólya (1957) notes the two extremes that can occur when devising a plan becomes difficult. "Some students rush into calculations and constructions without any plan or general idea; others wait clumsily for some idea to come and cannot do anything that would accelerate its coming" (p. 95).

It is helpful to note some of the mistakes that students make when solving the problem, and an example is given in **figure 3**. The student first tries to apply the Pythagorean theorem, with the 9/4 ratio acting as the hypotenuse length, but then crosses out that attempt. Instead, she simply decides to add the 9/4 to the vertical leg to get the hypotenuse. Even though the question asks for the length of the ladder, the student then goes on to use the Pythagorean theorem to calculate the missing leg from the hypotenuse and the vertical leg. Other examples of student mistakes

Fig. 3 The following student work is an incorrect solution to the Leaning Ladder problem.

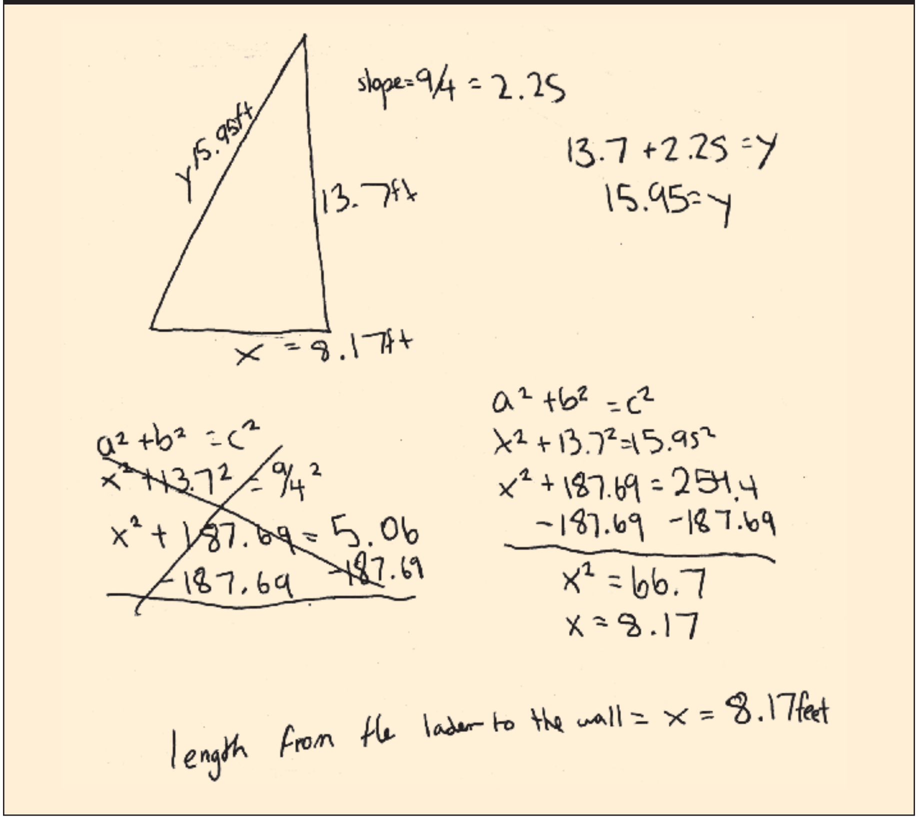


Fig. 4 Multileg Trip problem

Problem 2. An author goes on a signing tour at local bookstores to help sell her new book. Starting at her home, she drives 90 miles north to Barnes and Noble Booksellers. She then drives 40 miles west to Borders Books. Next she drives 150 miles south to Brentano's Bookstore. Finally, she drives 85 miles east to Crown Books. How far is the author from her home?

Content connection:
Integer addition in the plane

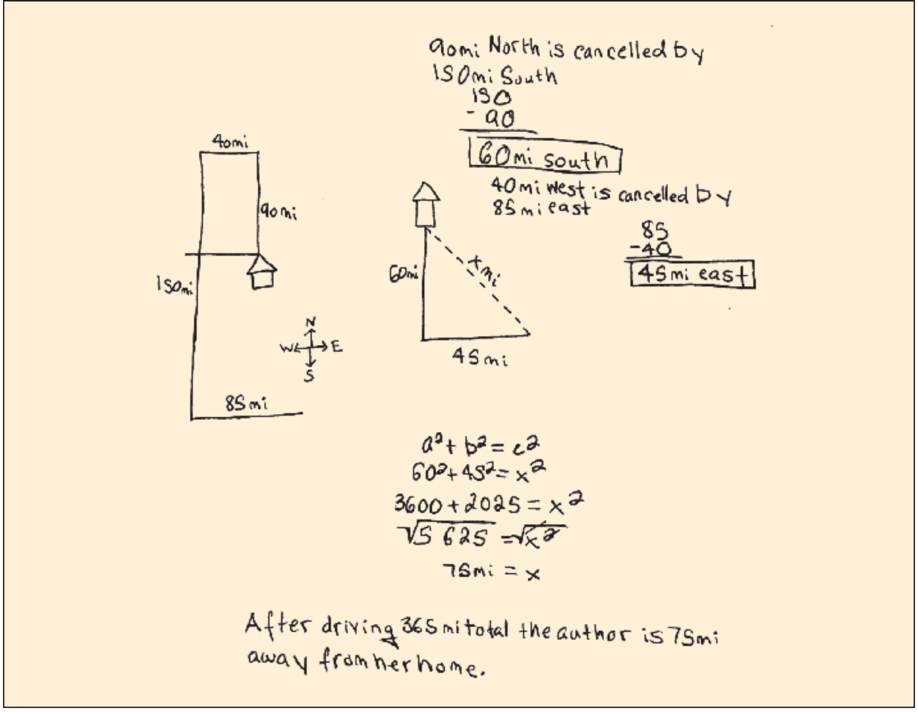


Fig. 5 A correct student solution to the Multileg Trip problem

include applying the Pythagorean theorem to the slope's rise and run to get the ladder length ($4^2 + 9^2 = x^2$); multiplying the vertical leg's length by the slope and then not using the Pythagorean theorem; mixing up the rise and run when setting up a proportion; using the slope as the missing leg ($13.7^2 + 2.25^2 = x^2$); and not using a proportion but various proportion-like arguments to get an incorrect length for the missing leg. Although these sorts of mistakes are not prevalent among completed solutions of student group work, they are indicative of the false starts and meanderings that can occur as students sort through ideas and search for a solution plan. The mistakes also reflect a core observation by most problem-solving advocates, namely, that "knowing how to execute procedures does not ensure that students understand what they are doing" (Hiebert and Wearne 2003, p. 3). Robust understanding needs flexibility, the ability to make connections and to apply ideas in new circumstances.

ADDITIONAL EXAMPLES AND DIFFERENT PROBLEM-SOLVING STRATEGIES

Further examples are discussed briefly. The next problem is a variation on the traditional travel question in which there are only two legs of a journey, such as the legs of a right triangle. Problem 2 discusses several legs of a trip (see **fig. 4**). A fairly standard list of classic problem-solving strategies, and good summaries of these techniques, often accompany collections of problem-solving activities (MATHCOUNTS 2005; Scanlin, Goodnow, Hoogeboom 1988). The strategies include using guess and check; considering a simpler case; making a table, chart, or list; looking for patterns; using logical reasoning; making a diagram or model; using objects or acting out the problem; working backward; brainstorming; and using the process of elimination. For the Multileg Trip problem, a careful diagram is an essential part of the strategy, allowing students to visualize how four distances can be effectively reduced to two through integer addition in the plane (see **fig. 5**). The net result of the four legs of the trip is the author ending up 60 miles south and 45 miles east from home. A right triangle with legs of 60 and 45 miles can be formed, leading to a Pythagorean calculation of a 75 mile distance from home. As shown in **figure 6**, the emergence of a single right triangle to describe the distance from home is opaque to some students. Even after calculating that the author is 60 miles south and 45 miles east, some will simply add these two distances to get 105 miles. Though the large majority of students tend to get the correct answer in the end, group work generally leads to real discussion and debate before the single Pythagorean calculation is recognized.

Problem 3 would be a routine exercise for a high school student, but it is a problem-solving situation for a seventh-grade prealgebra student (see **fig. 7**). In contrast to the first two problems, the Circle Discovery problem is more open-ended and has multiple solutions (indeed, infinitely many). Though to get a complete solution is challenging, most students feel quite comfortable digging into the problem using guess and check, combined with making a list. The

Fig. 6 This student solution to the Multileg Trip problem is incorrect. Although it is not the primary mistake, the final sentence gives an addition reference (85 mi. + 40 mi.) that is inconsistent with the earlier addition of 60 + 45 = 105.

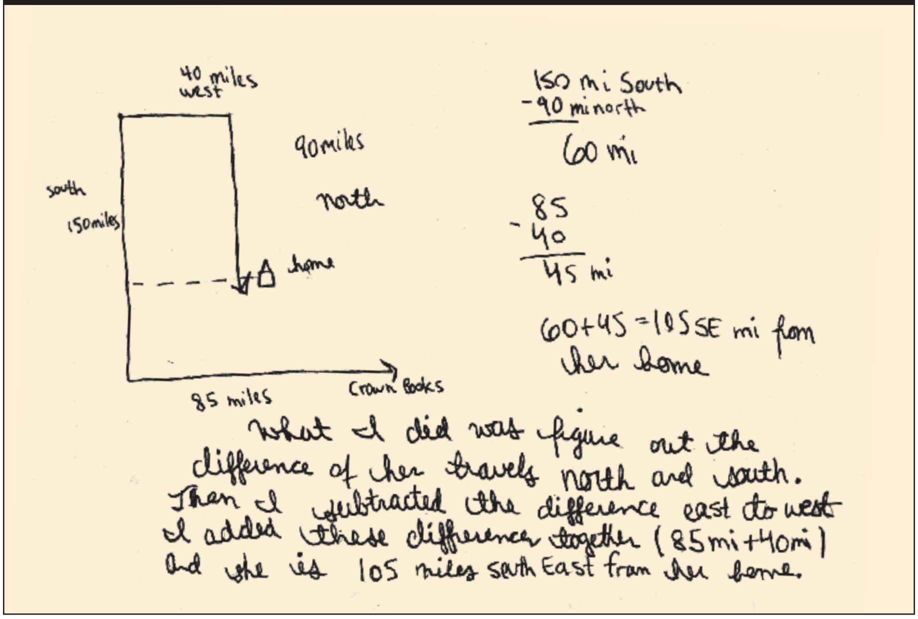


Fig. 7 Circle Discovery problem

Problem 3. Graph all ordered pairs (x, y) in the coordinate plane that satisfy the Pythagorean equation below.

$$x^2 + y^2 = 5^2$$

Content connection: Graphing

finding-a-pattern strategy eventually becomes part of the plan, as visual clues emerge from the developing graph. Hovering in the background is student familiarity with making a table of values, and I tend to hear, for reassurance, "Should we make a table?" As I watch student groups work on this problem, it is particularly interesting to see the evolution in their understanding, accompanied by a healthy dialogue of clarifying questions. Most students find the basic positive ordered pairs (5,0), (0,5), (4,3), and (3,4) fairly easily. Students start asking me whether negatives are allowed, and they soon define ordered pairs in other quadrants. Students also ask whether decimals are allowed, and I am impressed by the detail and thought that can accompany such questions. Referring to the case $x = 2$, some students observe, "We could have $2^2 = 4$, and then y would have to be $\sqrt{21}$. How do you plot that?" Most accept my explanation that $\sqrt{21}$ can be approximated by a decimal, but some remain uneasy about plotting approximate points on the graph. I ask them to write their best estimates of where the points should be and to see if a pattern emerges. A more pressing concern emerges when students realize that there are infinitely many solutions, and they exclaim, "This can take forever." They are asked to keep trying and to look for a pattern. Then they ask, "Should we connect the dots?" I ask them why we connect the dots with graphs of lines and whether that would make sense here. For problems like these, students are encouraged to interact only within their group and to avoid giving anything away prematurely to other groups. When students say, "Look! It's a circle!" it is obvious that they have made the discovery on their own (see **fig. 8**).

The Circle Discovery problem helps avoid having graphing become too mechanical for the students and forces them to revisit basic graphing concepts in a new situation—what it means to be a solution and how a graph is a visual representation of infinitely many solutions. In addition, the informal solution strategy leads to a new mathematical result of a circle graph and sets the stage

Fig. 8 A correct student solution to the Circle Discovery problem. Note that terms like -5^2 should have been written $(-5)^2$, a notational issue that will come up when the student takes formal algebra.

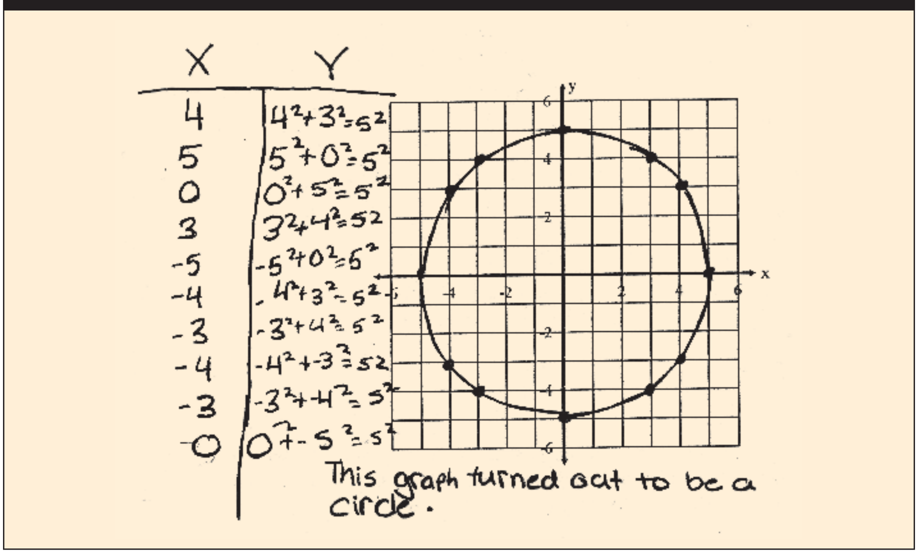


Fig. 9 Pythagorean Triplets problem

Problem 4. Give all examples you can determine of three positive integers satisfying the Pythagorean equation.

$$a^2 + b^2 = c^2$$

Content connection:
Number theory

for future algebraic thinking. In her call for change in mathematics instruction, Lindquist (1989) discusses the importance of student construction and ownership of content. “When students are using prior knowledge to construct new mathematical knowledge, they are learning mathematics. Otherwise, they are receiving a body of knowledge, and often in unrelated and unorganized pieces, which makes it difficult to retrieve and use” (p. 3). Schroeder and Lester (1989) highlight the dynamic interplay between problem solving and understanding, with each supporting the other. Just as “solving a problem can deepen a student’s understanding of a topic of mathematics” (p. 40), it is also true that “understanding promotes transfer of knowledge to related problems and its generalizability to other situations” (p. 41). In other words, the experience of discovering a circle graph helps students appreciate the connection between an algebraic equation and its graph, something that will be studied with increasing sophistication throughout the mathematics curriculum.

A search for Pythagorean triplets in problem 4 leads to a similar problem-solving experience (see **fig. 9**). Again we have infinitely many solutions that can be searched through guess and check and supported by an organized list. By far, the most common first result is a 3, 4, 5 triangle. The time it takes for students to get such a first result can vary greatly. As additional triplets are found, students discover the role of multiples in finding new combinations. Initially, students might not be able to phrase this discovery in words or a mathematical statement, but it guides their thinking as new triplets come quicker to them. Several students observe that their list could go on forever, and I ask them to keep going until they find a pattern, then describe how the list would continue. Most make a statement involving multiples, and some are able to give an algebraic equation (see **fig. 10**). Again, an informal solution strategy leads to new mathematics. One of the most significant first algebraic experiences occurs when students learn to make algebraic expressions and equations to describe patterns, rather than simply learn the mechanics of how to solve a given equation. A good collection of number patterns leading to algebraic formulas is given in Phillips (1991).

We offer four more problems, without student work and with minimal discussion, to convey to the reader the robustness of connecting different content areas in mathematics. Integrating content does not have to be an occasional curiosity but something that can be pursued regularly and vigorously by the teacher. Problem 5 asks students to identify nested similar triangles to invoke a similarity argument (see **fig. 11**). The students also need to break down line segments into different pieces applying to different triangles. Using the Pythagorean theorem to calculate the horizontal leg of the inner triangle (with a 30-inch hypotenuse) is a common solution. Then y is calculated using a proportion equation following from similarity

Fig. 10 A student solution to the Pythagorean triplets problem. The solution is representative of the fact that students almost always get only the triplets that are multiples of 3, 4, 5 triangles (as opposed to, say, a 5, 12, 13 triangle).

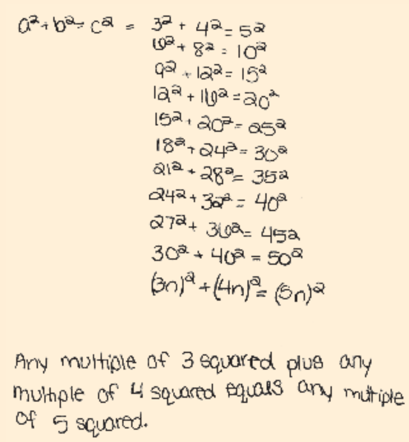
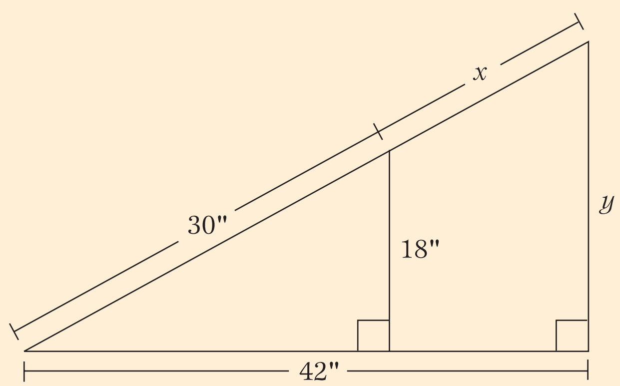


Fig. 11 Nested Similar Triangles problem (Answer: $x = 22.5$ in., $y = 31.5$ in.)

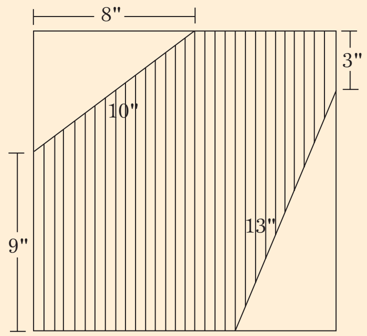
Problem 5. Find the missing lengths x and y.



Content connection: Similarity

Fig. 12 Geometric Probability problem (Answer: 57/75, or 76 percent)

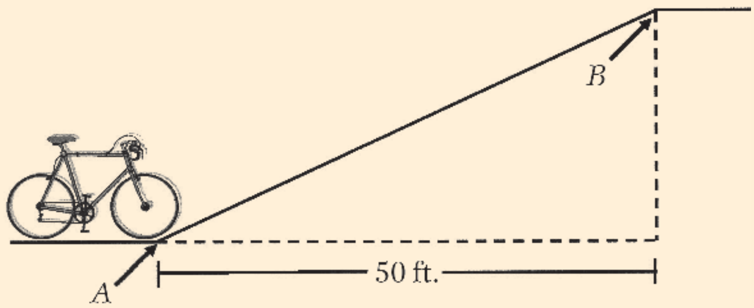
Problem 6. A dart is thrown randomly and lands on the square board. What is the probability that the dart will land in the shaded region?



Content connection: Probability

Fig. 13 Cycling Up a Ramp problem (Rounded answer: 26.4 ft.)

Problem 7. When the bicycle's front wheel travels from point A to point B, it does exactly 8 revolutions. If the tire diameter is 27 inches, how high is the top of the ramp from the ground?



Content connection: Circles

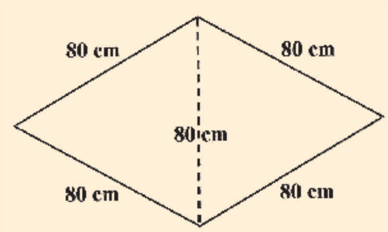
between the smaller triangle and the larger one in which it is embedded. Finally, x is calculated by applying the Pythagorean theorem to find the hypotenuse of the larger triangle (x + 30). Sandwiching a similarity argument between two Pythagorean calculations can produce an involved solution.

Problem 6 uses area calculations to solve a geometric probability question (see **fig. 12**). The calculation is more manageable when students recognize that the shaded area is the difference between the total area of the square board and the unshaded area. The two unshaded right triangles are missing one or two leg lengths, but these lengths can be found using a combination of Pythagorean calculations. The next two problems are by design more manageable, for we want to impress how asking students to integrate content and make decisions about solutions need not be overly difficult. In problem 7, the hypotenuse length is calculated as 8 tire circumferences before applying the Pythagorean theorem (see **fig. 13**). In problem 8, each equilateral triangle is divided into two right triangles to make a Pythagorean calculation of the height of the equilateral triangles (see **fig. 14**). Plenty of additional problems are given to the students, making connections to other content topics. Pythagorean calculations are embedded in volume and surface area calculations of prisms, pyramids, and cones; percent changes in lengths are studied to determine resulting percent changes in areas of two-dimensional shapes; nontrivial fractional calculations are incorporated in Pythagorean calculations; and so on.

The Pythagorean theorem has been the central topic of our content connections, but our goal is to encourage open-ended integration of diverse content areas throughout the prealgebra curriculum. My interest in such integration was sparked by impatience with routine exercises in my first years of teaching, coupled with a firm commitment to having my students master a solid content base. I found myself sprinkling in problems that connected different content areas in the hope of fostering critical thinking while reinforcing the curriculum. Incorporating such nonroutine problems on a regular basis is a natural way for teachers to have their students become more invested in the problem-solving process. As students become more comfortable with risk-taking and problem solving, nonroutine problems can take a more prominent role, including grouping together collections of problems to form theme-based activities such as our Pythagorean study.

Fig. 14 Kite Construction problem (Rounded answer: 5543 sq. cm.)

Problem 8. Bobby builds a kite by joining together two equilateral triangles with side lengths of 80 centimeters. What is the area of the kite?



Content connection: Measurement

CONCLUSION

Questions that integrate mathematical content from across the curriculum provide opportunities for students to develop a broader and more flexible understanding of the material. Having students make connections between distinct topics and procedures, rather than simply executing previously learned algorithms in a familiar setting, results in the decision-making process that is a trademark of problem solving. Some of our Pythagorean questions invoke standard problem-solving strategies, such as constructing a diagram, making an organized list or chart, and looking for a pattern. However, some questions, like the Leaning Ladder problem and the Nested Similar Triangle problem, do not use any of the classic strategies we mentioned earlier: We do not work backward, we do not consider a simpler case, we do not use models, and so on. These problems are generally challenging and require that students make real decisions to construct a solution plan. We would characterize the problem-solving strategy in these cases by the word *synthesis*—the combining of distinct parts to form a whole. The parts are content-rich specific procedures: using slope to solve a proportion, adding integers in the plane, invoking similarity in nested triangles, and quantifying probability through differences and ratios of areas. The synthesis of these parts into a whole is the problem-solving activity that is embedded in curriculum studied throughout the year. Whenever possible, we steer the direction of problem solving into the construction of new ideas and topics, such as the creation of a circle graph and the discovery of an algebraic pattern for Pythagorean triplets. When students connect ideas and create new ones, they have greater ownership of the learning process.

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